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Performance and Stability Margins Comparison of Pole Placement and Optimal Controllers using Cart-Inverted Pendulum System

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***Abstract*— In this paper, pole placement and two optimal control techniques which are the linear quadratic regulator and linear quadratic Gaussian are compared. A cart and inverted pendulum which is an inherently unstable dynamical system is used as a case study to analyze their performance and stability margins. Lagrangian equations defining the system dynamics are converted to linear state-space representation. The objective is to keep the pendulum in an upright position as the cart on which it is mounted moves from one position to another. MATLAB is used to solve the optimization problem and simulate the step response of the system. The robustness of both controllers is measured by giving uncertain model parameters to the system and observing the level of uncertainty these controllers can handle. The simulation results justify the relative advantages of these control schemes.**

Index Terms—Inverted pendulum, Riccati equations, state space model, linear quadratic regulator (LQR), linear quadratic Gaussian (LQG), Kalman filter

I. INTRODUCTION

Inverted pendulum is a highly non-linear and unstable system. The control of cart and inverted pendulum is a difficult task since it belongs to a class of under actuated systems. Inverted pendulum system is a classical problem in control engineering and has been modelled and studied for testing and performance comparison of different control techniques [1]. The aim of this paper is to study stability of two optimal controllers and compare their stability margins when the system is subjected to uncertain model parameters. In this paper pole placement, linear quadratic regulator and linear quadratic Gaussian methods are employed to stabilize the system. LQR minimizes a cost function which is based on weights of states and control inputs. LQG is an extension of LQR where an additional state estimator is used. Commonly used estimator is Kalman Filter [7]. It is used to estimate the critical states of the system based on the given sensor measurements. Kalman filter also rejects any disturbances and noise in the system or sensor measurements. Often optimization problem is solved using some computational tool such as Python or MATLAB. LQR and LQG methods requires the system to be controllable and observable [9]. This paper is organized as follows. In section II, system is modelled using Langrage equations. Section III presents the controller design techniques. In section IV MATLAB simulation results are presented along with discussion on the settling time and stability margins. Finally, section V is dedicated to conclusion.

II. SYSTEM MODELLING

The dynamic system consists of cart of mass M and pendulum of mass m, pendulum is free to rotate about its hinge and cart can translate in x and -x direction, this system is modelled as non-linear 2nd order differential equations which are then linearized about $\theta=\pi$ [2]. The cart is acted upon by an external force F and the goal is to keep the pendulum in vertical position as the cart moves from zero to one position. In its natural response pendulum falls from its vertical position and oscillates in downward position. The cart pendulum system is shown in the following Figure 1.

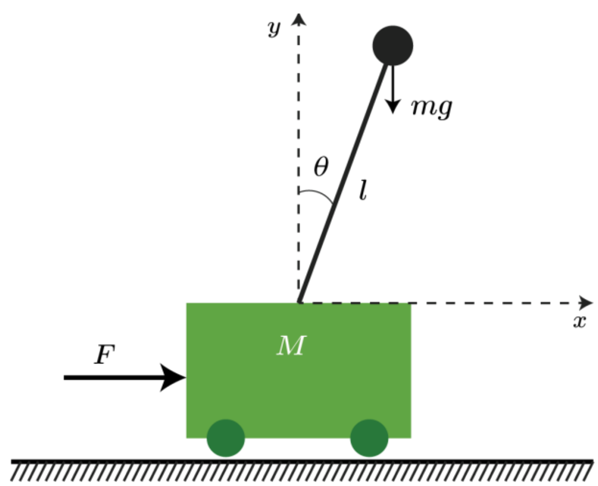

Fig. 1 Cart and Inverted Pendulum

The model parameters are shown in the following Table -1

Table 1 System Parameters

Parameters	Values
Pendulum mass "m"	0.4 kg
Friction coefficient "b"	0.3
Pendulum center length l	0.5 m
Moment of inertia I	0.0083 m^4
Cart mass "M"	1 kg

In the horizontal direction, sum of all forces as shown in the free body diagram gives the first governing equation (1) of motion:

$$(M+m)x'' + bx' - ml\theta'^2 \sin\theta + ml\theta'' \cos\theta = F \quad (1)$$

In the vertical direction, sum of all forces gives the second governing equation (2):

$$mgl \sin\theta + (I+ml^2)\theta'' = -mlx'' \cos\theta \quad (2)$$

These equations are linearized about vertically upward pendulum position $\theta = \pi$ and using small angle approximation [8]. After the input force F is substituted by u the system dynamics become (3):

$$\left(I+ml^2\right)\theta^2 - mgl\theta = mlx'' \quad (3)$$

and(4):

$$(M+m)x'' + bx' - ml\theta'' = u \quad (4)$$

These linearized equations can be expressed in standard linear state space form, for that state variables are defined as below:

$$x_1 = \theta$$

$$x_2 = \theta'$$

$$x_3 = x$$

$$x_4 = x'$$

Since the standard linear time invariant state space representation has the following form:

$$x' = Ax + Bu \quad (5)$$

$$y = Cx + Du \quad (6)$$

Open loop response

Our system in state space representation can be written as following form.

$$\begin{bmatrix} x' \\ x'' \\ \theta' \\ \theta'' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} \\ 0 & 0 & 0 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mm} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(I+ml^2)}{I(M+m)+Mml^2} \\ 0 \\ \frac{ml}{I(M+m)+Mml^2} \end{bmatrix} [u] \quad (7)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \quad (8)$$

Eigenvalues of matrix A are the open loop poles of the system.

$$0, \ -4.9846, \ -0.1428, \ 4.9334$$

With these eigenvalues, open loop response of the system is plotted in the Figure 2.

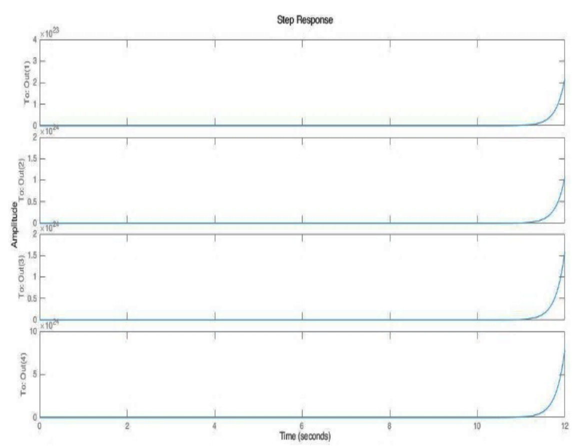

Fig. 2 Open Loop Response

III. CONTROLLER DESIGN

As, it is evident, one of the system poles lies on right side of complex plane therefore the open loop system is unstable. Here three techniques will be employed to design the controller to stabilize the system. Before starting to design the optimal controller it's important to know if the system is controllable and observable [4]. Controllability determines if there are enough inputs to the system so that it's all critical states can be brought to any desired value by given inputs within a finite amount of time. Mathematically a system is controllable if matrix P is full rank.

$$P = [B \;\; BA \;\; BA^2 \;\; BA^3 \ldots, BA^{n-1}] \qquad (9)$$

By observable system, it is meant that critical states that are not directly measured can be estimated using only few measured output states y [10]. Mathematically for a system to be observable the matrix O should have full rank.

$$O = [C \;\; CA \;\; CA^2 \ldots, CA^{n-1}]^T \qquad (10)$$

The ranks of these matrices were calculated and were found to be equal to 4, thus both conditions were satisfied.

A. *Linear Quadratic Regulator*

The linear quadratic regulator solves the optimization problem using system and weighting matrices to calculate the gain matrix K that stabilizes the system. The objective of the LQR method is to minimize the cost function J based on Q and R matrices [11]. The cost function is given by:

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \qquad (11)$$

The Q and R matrices are defined as:

$$Q = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R = [0.01]$$

State feedback $u = -Kx$ minimizes the J cost function.

The gain matrix K is found by using (12):

$$K = R^{-1} B^T S \qquad (12)$$

Here S is symmetric matrix found by solving algebraic Riccati equation(13):

$$A^T S + SA - SBR^{-1}B^T S + Q = 0 \qquad (13)$$

The block diagram of the system with the LQR is shown in Figure 3.

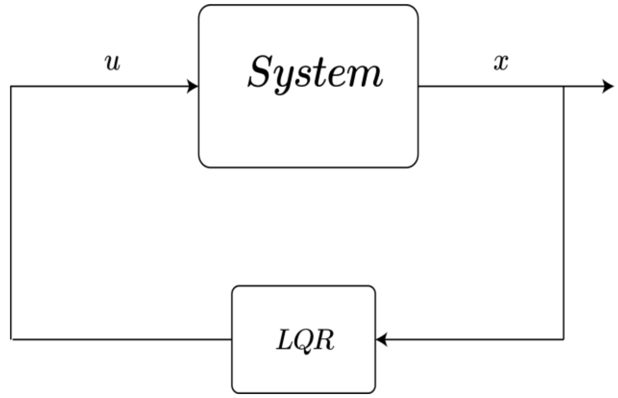

Fig. 3 Linear quadratic regulator

B. *Linear Quadratic Gaussian*

In practical scenarios often it is difficult to measure all the states at the output, in such cases an estimator is used that takes the measurements y and estimates the full state of the system, given that the system is observable. This is called full state estimation or linear quadratic Gaussian. Kalman filter is often used as an observer [7]. Kalman filter takes system input u and measured state y as input and gives $\hat{x}$ as output. This dynamic system filters out any disturbances and noise in the system, gives the optimal estimate of states $\hat{x}$ by minimizing variance of the variable measured [3]. In this study, cart and inverted pendulum system was subjected to a white noise of the following form Figure 4.

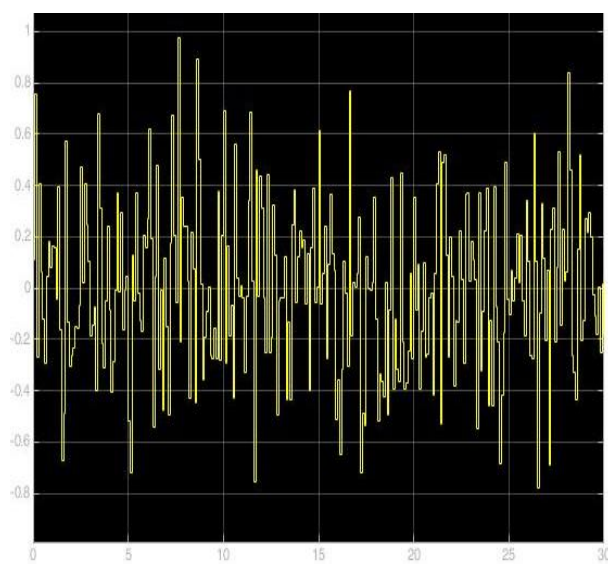

Fig. 4 White noise

The state estimation by Kalman filter is given by (14) and (15) equations:

$$\hat{x}' = (A - K_f C)\hat{x} + Bu + K_f y \qquad (14)$$

$$\hat{y} = Cx + Wn \qquad (15)$$

In case of LQG, system in state space is represented as:

$$x' = Ax + Bu + W_d \qquad (16)$$

$$y = Cx + Du + W_n \qquad (17)$$

Difference between actual and measured states is given as:

$$e = x - x^{\wedge} \qquad (18)$$

$$e' = \left(A - K_f C\right)e + W_d - K_f W_n \qquad (19)$$

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK_f & BK_r \\ 0 & A - K_f C \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} I & 0 \\ I & -K_f \end{bmatrix}\begin{bmatrix} W_n \\ W_n \end{bmatrix} \qquad (20)$$

The state of the system is still stabilized by LQR eigenvalues or $A - BK_f$. The block diagram of the system with full state estimation and LQR is shown in Figure 5.

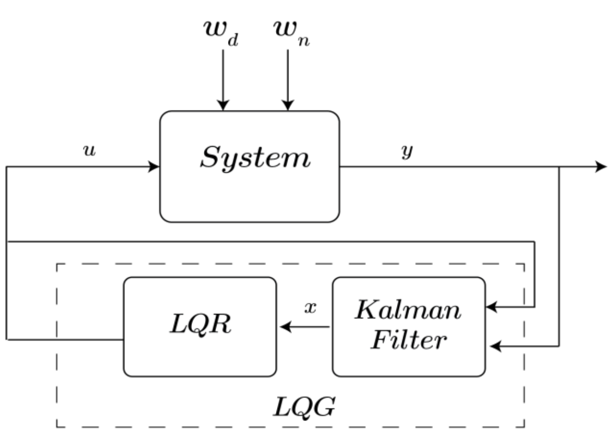

Fig. 5 Linear quadratic Gaussian

C. Pole Placement

Given that system is controllable and observable [6], it is intuitive to think that by arbitrarily placing the poles in left half of the complex plane should stabilize the system. Keeping this in consideration four random eigenvalues were chosen and gain matrix K was computed for the system. These eigenvalues determine the response of the system.

IV. Results

Linear quadratic regulator places the closed loop poles at -6.7469± 1.8446i, -2.3681 ± 1.5602i and the system with feedback has a step response Figure 6:

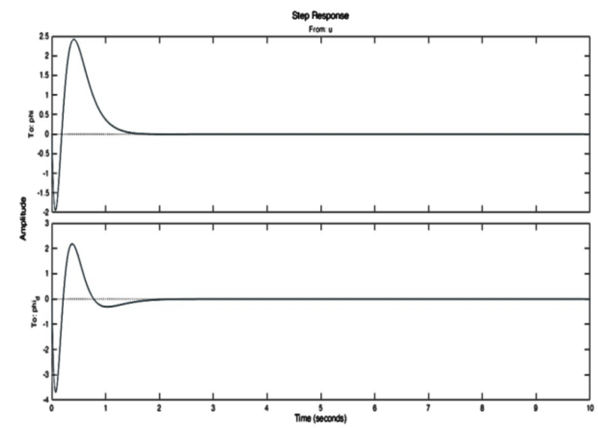

Fig. 6 Step response of the system with LQR

For LQG controller, eigenvalues of the estimator such as Kalman filter determine how aggressively or slowly Xˆ will converge to X. The closed loop poles of the system are -0.8778 ± 0.5002i, -4.8815 + 0.0000i, -5.0333+ 0.0000i, and step response of the system with linear quadratic estimator is Figure 7:

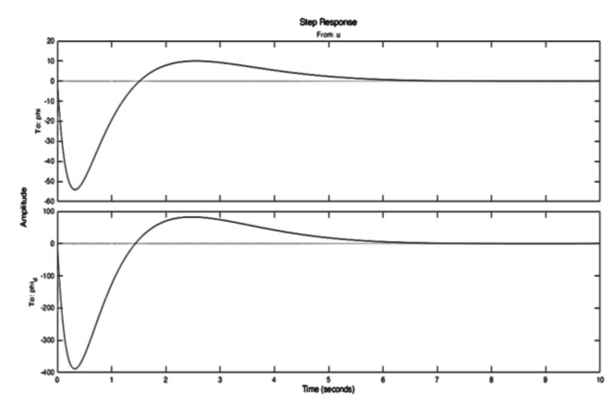

Fig. 7 Step response of the system with LQG

The closed loop poles are chosen arbitrarily as: -1,-1.2,-1.4,-1.6. The step response of the system in case of pole placement technique is shown in Figure 8:

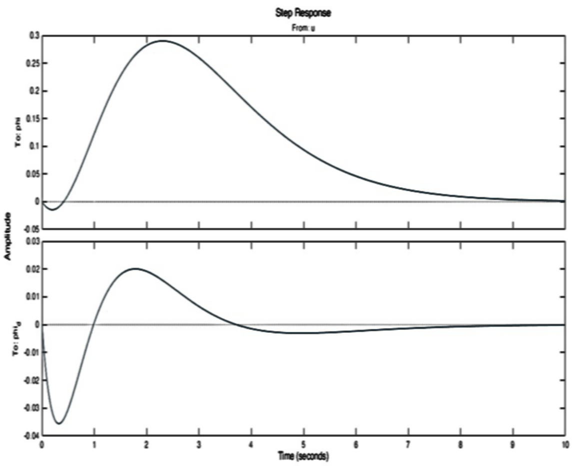

Fig. 8 Step response of the system with Pole Placement

The settling time of theta from vertical position for controllers LQR and LQG is 1.92 and 5.72 seconds respectively. Stability margins of the closed loop systems are found by varying the model parameters from their nominal values using MATLAB. System parameters were varied up to 20%. They are direct measurement of the robustness of the control system. In case of LQR control, system is robustly stable and can tolerate up to 2.7 times the modelled uncertainty. The smallest values of parameters for worst case gains that can cause instability in the system are M = 1.5395, l = 0.7697, m = 0.1842. In case of LQG control, the system is not robustly stable for even 10% of the uncertain model parameters. It can take only up to 6.22% parameter variations in the system. For the case of pole placement method, system was found to be robustly stable for the 20% modelled uncertainty. Also, it was found that it can tolerate up to 5.45 times of the 20% uncertain parameters. At 121% of the modelled uncertainty a perturbation makes the system unstable at the frequency of 1.29 rad/s. Mass of the cart was found to be the most critical parameter which has highest impact to decrease the stability margins of the system.

V. CONCLUSION

The problem was to keep the pendulum in vertical position as the cart moves from one position to another. Open loop system was unstable and required a feedback control to have a stable response. For this, two optimal control design approaches were used and were compared. LQR has smaller settling time and can take much higher system variations before it goes unstable. However, that’s does not mean its “superior” strategy over LQG method. It should be noted that there existed system disturbances and sensor noise in case of LQG and not all states were measured at the output of the system. The only information at the output was position of the cart from which other states were estimated. Linear quadratic regulator controller should be used where we have all the information of the states at the output. Also, it can be said that it’s more robust than linear quadratic Gaussian as it handles most model uncertainties. On the contrary, LQR with Kalman filter should be used where all system states are not known at the output. Moreover, in case of LQG, future uncertainties should be modelled in the system as it can be highly sensitive to model uncertainties. It can also be said that LQR is more robust and LQG is more practical comparatively. Finally, it should be made clear that though optimal controllers give room to choose weighting Q and R matrices, but this can be sometimes dangerous as higher values may even break the system.

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